Complex analysis for EE, 2012-13, problem set 7

1. Suppose a function f is holomorphic on $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}$. Show that there exist entire functions $f_0, f_1, \ldots, f_n$ such that $f(z) = f_0(z) + f_1(\frac{1}{z-z_1}) + \ldots + f_n(\frac{1}{z-z_n})$.
Solitions:
We do is by induction on n. For $n = 0$ there's nothing to prove.
Now, suppose the claim holds for $n \geq 0$, and let f be holomorphic on $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_{n+1}\}$ (for some arbitrary collection of $n+1$ distinct points of singularity). Note that due to the finite number of singularities, they are all isolated. Examine
$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_{n+1})^m + \sum_{k=1}^{\infty} \frac{b_k}{(z - z_{n+1})^k},$$
the Laurent series expression of f about z_{n+1}. Note that it is holomorphic in a punctured disc (a ring with zero inner radius), hence f_{n+1} defined $f_{n+1}(w) = \sum_{k=1}^{\infty} b_k w^k$ is entire. Also, note that $g(z) = f(z) - f_{n+1}(\frac{1}{z-z_{n+1}})$ is holomorphic in $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}$, so by the induction assumption we're done.

2. Suppose f is holomorphic in $\mathbb{C}$, except perhaps at finitely many poles, and either has a pole at infinity or the limit $\lim_{z\to\infty} f(z)$ exists. Such a function is called *meromorphic* on the Riemann sphere. Show that f is a rational function (hint: isolate the poles of f to arrive by an entire function g such that $f(z) = g(z) / \prod_{j=1}^{n} (z - z_j)^{m_j}$. Next, what can you say about g's zeroes?)
Solitions:
We begin by assuming without loss of generality that $\lim_{z\to\infty} f(z) \neq 0$ (if it is zero, then take $h(z) = f(\frac{1}{z})$), which is holomorphic in a neighborhood of 0 with $h(0) = 0$. Either h is identically zero in that neighborhood–implying that f is zero everywhere–or there exists $n \in \mathbb{N}$ such that $\lim_{z\to 0} \frac{h(z)}{z^n} = w_0 \neq 0$. In the latter case, we show that $f(z)z^n$ is rational, and therefore f is as well).

Next, we recall that f has a pole of degree m at z_0 if and only if $f(z) = \frac{g(z)}{(z-z_0)^m}$ where g is holomorphic in a neighborhood of z_0. We use induction to prove that we can represent
$$f(z) = \frac{g(z)}{\prod_{j=1}^{n} (z - z_j)^{m_j}},$$
where g is entire, and for all $1 \leq j \leq n$, z_j is a pole of degree m_j for f. For $n = 0$, there's nothing to prove. Suppose the claim holds for $n \in \mathbb{N}$, and let f satisfy the conditions of the exercise with $n+1$ poles, at $\{z_1, z_2, \ldots, z_{n+1}\}$. We therefore have $\tilde{f}(z) = (z - z_{n+1})^{m_{n+1}} f(z)$ which is holomorphic on $\mathbb{C} \setminus \{z_j\}_{j=1}^{n}$. By induction, we get an entire $g(z) = \tilde{f}(z) \prod_{j=1}^{n} (z - z_j)^{m_j} = f(z) \prod_{j=1}^{n+1} (z - z_j)^{m_j}$, and we're done.

Note that g is entire, and $\lim_{z\to\infty} g(z) = \infty$. It will therefore suffice to reduce the question into the following, special case: suppose g is entire and has a pole at infinity. Then g is a polynomial.

For such g, examine the Taylor series about zero $g(z) = \sum_{n=0}^{\infty} a_n z^n$. We know that $h(z) = g(\frac{1}{z})$ has a pole at zero, but by substitution its Laurent series about zero is $h(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$, implying that $a_n = 0$ for all $n > N$, for some $N \in \mathbb{N}$. Therefore, g is polynomial.

3. Let Ω be a bounded region, and let $F \subseteq \Omega$ be closed. Show that there does not exist a function f with infinitely many poles in F that otherwise holomorphic on Ω.

__Solitions__:

We show that if $P \subseteq F$ is some subgroup, and each $p \in P$ is a pole a holomorphic $f : \Omega \setminus P \to \mathbb{C}$, then P has finitely many points.

Ω is bounded, hence so is F. It follows that F is compact, and therefore P has an accumulation point in F (that is, a point $w \in F$ such that for all $\varepsilon > 0$, $D'_\varepsilon(w) \cap P \neq \emptyset$. You might be more familiar with the following phrasing: every bounded sequence in $\mathbb{C}$ has a converging subsequence). We know that $w \notin P$, because it wouldn't be isolated (poles are always isolated singularities). Therefore, f is differentiable at w, but f isn't even continuous at w (it assumes arbitrarily large values in every neighborhood of w), in contradiction.

4. Find the residues at the poles of the following functions:

   (a) $\frac{1}{z^3(z^2+1)}$

   __Solitions__:

   We can easily find the Laurent series in $D'_1(0)$. For $0 < |z| < 1$, one notes

   $$\frac{1}{z^3(z^2+1)} = \frac{1}{z^3}\sum_{n\geq 0}(-1)^n z^{2n} = \frac{1}{z^3} - \frac{1}{z} + z - \dots,$$

   Hence the residue at the origin is (-1).

   For $z = i$, we conveniently denote $\frac{1}{z^3(z^2+1)} = \frac{\frac{1}{z^3(z+i))}}{z-i}$, implying that the residue at $z = i$ equals $\frac{1}{i^3(i+i))} = \frac{1}{2}$. Similarly, the residue at $z = -i$ equals $\frac{1}{2}$ as well.

   (b) $\frac{1}{z(3-z)}$

   __Solitions__:

   Here, the residue at $z = 0$ equals $\frac{1}{3-0} = \frac{1}{3}$, and the residue at $z = 3$ equals $\frac{-1}{3} = -\frac{1}{3}$.

   (c) $\frac{1}{1-z+z^2}$

   __Solitions__:

   We note that $\frac{1}{1-z+z^2} = \frac{4}{(2z-1-i\sqrt{3})(2z-1+i\sqrt{3})}$, so the residue at $z_0 = \frac{1+i\sqrt{3}}{2}$ equals $\frac{4}{2\cdot 2i\sqrt{3}} = -\frac{i}{\sqrt{3}}$, and the reside at $\overline{z_0}$ equals $\frac{i}{\sqrt{3}}$.

   (d) $\frac{1}{(z^2+z+1)^3}$

   __Solitions__:

   Relying on the last part, we know that the residue at z_0 equals

   $$\frac{1}{2!}\lim_{z\to z_0}\left((z+z_0)^{-3}\right)^{(2)} = 6(2z_0)^{-5}.$$

   Similarly, the residue at $\overline{z_0}$ equals $-6(2z_0)^{-5}$.

   (e) $\frac{e^{iz}}{\cosh(z)}$

   __Solitions__:

   $\cosh(z) = 0$ only at $(k+\frac{1}{2})\pi i$ for some $k \in \mathbb{Z}$. At each such point, the pole is of degree one, and the residue is $e^{i(k+\frac{1}{2})\pi i} = \frac{e^{-(k+\frac{1}{2})\pi}}{\sin((k+\frac{1}{2})\pi)} = (-1)^k e^{-(k+\frac{1}{2})\pi}$.

   (f) $\frac{1}{(z^2+1)\sin(\pi z)}$

   __Solitions__:

   Again, the function only has simple poles, at $z = \pm i$ and $z = k \in \mathbb{Z}$. The residue at $z = i$ is $\frac{1}{2i\sin(\pi i)}$, and the same at $z = -i$. The residue at $z = k \in \mathbb{Z}$ is $\frac{1}{\pi(k^2+1)\cos(\pi k)} = \frac{(-1)^k}{\pi(k^2+1)}$.

(g) $\frac{1}{z^4\sin(z)}$
Solitions:
This function has simple poles at $z=\pi k$ for $k\in\mathbb{Z}$, with residue $\frac{(-1)^k}{\pi^4 k^4}$. It also has a pole of degree 5 at zero. To find the residue there, we can of course calculate $\frac{1}{4!}\lim_{z\to 0}\left(\frac{z}{\sin z}\right)^{(4)}$, but an easier approach exists. Note that $\frac{1}{z^4\sin(z)}$ has a Laurent expansion in $D_1'(0)$, which we can find in the following way:

$$\frac{1}{z^4\sin(z)}=\frac{1}{z^5\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-z^6(\dots)\right)}.$$

We easily verify that each addend is smaller in absolute value than its predecessor, with alternating signs, so each addend is larger than the remaining tail of the series. In particular, $\left|\frac{z^2}{3!}-\frac{z^4}{5!}+z^6(\dots)\right|<1$. Therefore we know that:

$$\begin{aligned}\frac{1}{z^4\sin(z)}&=\frac{1}{z^5}\left(1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+z^6(\dots)\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+z^6(\dots)\right)^2+\dots\right)=\\&=\frac{1}{z^5}\left(1+\frac{1}{3!}z^2+\left(\frac{1}{3!^2}-\frac{1}{5!}\right)z^4+z^6(\dots)\right).\end{aligned}$$

It follows that the residue at zero equals $\left(\frac{1}{3!^2}-\frac{1}{5!}\right)=\frac{7}{360}$.

(h) $\tan(z)^2$
Solitions:
We note that $\tan^2 z=\frac{1}{\cos^2 z}-1=(\tan z-z)'$. It follows that the residue at each zero of $\cos z$ is zero.

5. By converting each integral into one along the unit circle, prove the following identities:

(a) $\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}d\theta=\frac{\pi}{4}$
Solitions:
We note that $5+4\cos\theta=1+4\cos\theta+4\cos^2\theta+4\sin^2\theta=\left|1+2e^{i\theta}\right|^2$, and also $\sin^2\theta=-\frac{1}{2}\operatorname{Im}\left(e^{i\theta}\right)\operatorname{Im}\left(1+2e^{-i\theta}\right)$. Hence:

$$\begin{aligned}\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}d\theta&=-\frac{1}{2}\int_0^{2\pi}\frac{\operatorname{Im}\left(e^{i\theta}\right)\operatorname{Im}\left(1+2e^{-i\theta}\right)}{\left|1+2e^{i\theta}\right|^2}d\theta=\\&=\operatorname{Im}\left(-\frac{1}{2}\int_0^{2\pi}\frac{\operatorname{Im}\left(e^{i\theta}\right)\left(1+2e^{-i\theta}\right)}{\left|1+2e^{i\theta}\right|^2}d\theta\right)=\\&=\operatorname{Im}\left(-\frac{1}{2i}\int_0^{2\pi}\frac{\operatorname{Im}\left(e^{i\theta}\right)}{1+2e^{i\theta}}e^{-i\theta}ie^{i\theta}d\theta\right)=\\&=\operatorname{Im}\left(-\frac{\pi}{2\pi i}\int_{|z|=1}\frac{z-\overline{z}}{2iz(1+2z)}dz\right)=\frac{\pi}{4}\operatorname{Re}\left(\frac{1}{2\pi i}\int_{|z|=1}\frac{z-1/z}{z(z+1/2)}dz\right)=\\&=\frac{\pi}{4}\operatorname{Re}\left(\frac{1}{2\pi i}\int_{|z|=1}\frac{z^2-1}{z^2(z+1/2)}dz\right)=\\&=\frac{\pi}{4}\operatorname{Re}\left(\operatorname{Res}\left(\frac{z^2-1}{z^2(z+1/2)},0\right)+\frac{z^2-1}{z^2(z+1/2)},-\frac{1}{2})\right)=\frac{\pi}{4}(4-3)\end{aligned}$$

(b) $\int_0^{2\pi} \frac{1}{1+a\cos\theta} d\theta = \frac{2\pi}{\sqrt{1-a^2}}$, where $-1 < a < 1$
Solitions:

$$\int_0^{2\pi} \frac{1}{1+a\cos\theta} d\theta = \int_0^{2\pi} \frac{2}{2+a(e^{i\theta}+e^{-i\theta})} d\theta = \frac{1}{i}\int_0^{2\pi} \frac{2}{2e^{i\theta}+a(e^{2i\theta}+1)} ie^{i\theta} d\theta =$$
$$= \frac{2}{i}\int_{|z|=1} \frac{dz}{2z+az^2+a} = \frac{2}{i}\int_{|z|=1} \frac{dz}{(az+1+\sqrt{1-a^2})(z-\frac{\sqrt{1-a^2}-1}{a})}$$

We note that $\left|-\frac{1+\sqrt{1-a^2}}{a}\right| > 1$, whereas $-1 < \frac{\sqrt{1-a^2}-1}{a} < 1$, hence $\int_0^{2\pi} \frac{1}{1+a\cos\theta} d\theta = \frac{4\pi}{2\sqrt{1-a^2}}$

(c) $\int_0^{2\pi} \frac{1}{(1+2a\cos\theta+a^2)^2} d\theta = \frac{2\pi(1+a^2))}{(1-a^2)^{3/2}}$, where $-1 < a < 1$
Solitions:

$$\int_0^{2\pi} \frac{1}{(1+2a\cos\theta+a^2)^2} d\theta = -i\int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}(1+ae^{i\theta}+ae^{-i\theta}+a^2)^2} d\theta =$$
$$= 2\pi\frac{1}{2\pi i}\int_{|z|=1} \frac{zdz}{(az^2+(1+a^2)z+a)^2}$$

For $a \neq 0$, this function has two poles of order 2 each, but only one of those is in the unit disc. In case $a > 0$, denote $b = \frac{1+a^2}{2a} > 1$, and this pole is at $z_0 = -b+\sqrt{b^2-1}$. The residue equals

$$\lim_{z\to z_0} \left(\frac{z}{a^2\left(z-(-b-\sqrt{b^2-1})\right)^2}\right)' = \frac{b+\sqrt{b^2-1}-z_0}{a^2\left(b+\sqrt{b^2-1}+z_0\right)^3} = \frac{2b}{a^2\left(2\sqrt{b^2-1}\right)^3} =$$
$$= \frac{1+a^2}{(1-a^2)^{3/2}}$$

The same holds (with minor adjustments) for $a < 0$, and the case of $a = 0$ is trivial.

6. By integrating along a rectangle which rests upon the real axis, show that $\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} dx = \frac{\pi}{\sin(\pi\alpha)}$, where $0 < \alpha < 1$.
Solitions:
We first note that the function $\frac{e^{\alpha z}}{1+e^z}$ has a single pole in the rectangle with vertices at $-R, R, R+2\pi i, -R+2\pi i$, at $z = \pi i$. Moreover, its residue at this point equals $\frac{e^{\alpha\pi i}}{e^{\pi i}} = -e^{\alpha\pi i}$. Also note that

$$\left|\int_{[R,R+2\pi i]} \frac{e^{\alpha z}}{1+e^z} dz\right| \leq 2\pi e^{(\alpha-1)R} \underset{R\to\infty}{\longrightarrow} 0$$
$$\left|\int_{[-R+2\pi i,-R]} \frac{e^{\alpha z}}{1+e^z} dz\right| \leq 2\pi\frac{e^{-\alpha R}}{1-e^{-R}} \underset{R\to\infty}{\longrightarrow} 0$$
$$\int_{[R+2\pi i,-R+2\pi i]} \frac{e^{\alpha z}}{1+e^z} dz = \int_R^{-R} \frac{e^{\alpha(2\pi i+t)}}{1+e^{2\pi i+t}} dt = -e^{2\pi i\alpha}\int_{-R}^{R} \frac{e^{\alpha t}}{1+e^t} dt$$

Therefore we can let R tend to infinity:

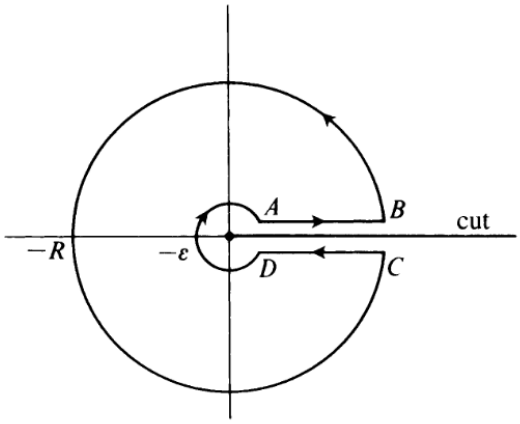
$$\int_{-\infty}^{\infty} \frac{e^{\alpha t}}{1+e^t} dt = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{\alpha t}}{1+e^t} dt = \frac{2\pi i(-e^{\alpha\pi i})}{1-e^{2\pi i\alpha}} = \frac{\pi}{\sin(\pi\alpha)}$$

7. Evaluate the following integrals

   (a) $\int_0^\infty \frac{dx}{x^{10}+x^5+1}$.
   Solitions:
   The roots of $z^2+z+1=0$ are $e^{\pm 2\pi i/3}$, therefore the roots of $z^{10}+z^5+1=0$ are $e^{2\pi i(\pm 1+3k)/15}$ for any $k \in \mathbb{Z}$. We integrate $f(z)=\frac{1}{z^{10}+z^5+1}$ along the slice of pie with vertices at $0, R, Re^{2\pi i/5}$ for sufficiently large $R>0$, in which f has poles at $e^{2\pi i/15}, e^{4\pi i/15}$, with residues $\frac{1}{-10e^{\pi i/5}+5e^{8\pi i/15}}, \frac{1}{10e^{2\pi i/5}-5e^{\pi i/15}}$, respectively. Finally, note that integrating f along the arc section of our contour vanishes at infinity:
   $$\left|\int_\frown f(z)dz\right| \leq \frac{2\pi}{5(R^10-R^5-1)} \xrightarrow[R\to\infty]{} 0.$$
   Therefore:
   $$\int_0^\infty f(x)dx = \frac{2\pi i(\mathrm{Res}(f, e^{2\pi i/15})+\mathrm{Res}(f, e^{4\pi i/15}))}{1-e^{2\pi i/5}}$$

   (b) $\int_{-\infty}^\infty \frac{e^{-i\omega x}}{x^4+x^3+x^2+x+1}dx$
   Solitions:
   We start by assuming $\omega<0$. We note that $f(z)=\frac{e^{-i\omega z}}{z^4+z^3+z^2+z+1}=\frac{(z-1)e^{-i\omega z}}{z^5-1}$. Therefore f has two poles in the upper half plane, at $z=e^{2\pi i/5}$ and $z=e^{4\pi i/5}$, with residues $\frac{(e^{2\pi i/5}-1)e^{-i\omega e^{2\pi i/5}}}{-5e^{3\pi i/5}}$ and $\frac{(e^{4\pi i/5}-1)e^{-i\omega e^{4\pi i/5}}}{-5e^{\pi i/5}}$, respectively. We integrate f along a half circle, and note that the integral along the arc vanishes:
   $$\left|\int_\frown f(z)dz\right| \leq \pi R\frac{(R+1)\sup_{0\leq\theta\leq\pi} e^{\omega R\sin\theta}}{R^5-1} = \pi R\frac{R+1}{R^5-1} \xrightarrow[R\to\infty]{} 0$$
   We can therefore calculate:
   $$\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_{-R}^R f(x)dx = 2\pi i\left(\mathrm{Res}(f, e^{2\pi i/5})+\mathrm{Res}(f, e^{4\pi i/5})\right)$$
   Finally, if $\omega>0$, we repeat the exercise with f's poles in the lower half plane.

8. A keyhole contour is described in the following image:



In this next exercise, we take $R=1$, and suppose f is holomorphic in some neighborhood of the closed unit disk. We take $\log z$ to be the analytic branch of logarithm in $\mathbb{C}\setminus[0,\infty)$.

(a) Show that

$$\int_{AB} f(z)(\log z - i\pi)dz = \int_{\varepsilon}^{1} f(x+i\delta)(\log(x+i\delta)-\pi i)dx \underset{\delta\to 0}{\longrightarrow} \int_{\varepsilon}^{1} f(x)(\log(x)-\pi i)dx,$$

and that

$$\int_{CD} f(z)(\log z - i\pi)dz = -\int_{\varepsilon}^{1} f(x-i\delta)(\log(x-i\delta)-\pi i)dx \underset{\delta\to 0}{\longrightarrow} -\int_{\varepsilon}^{1} f(x)(\log(x)+\pi i)dx.$$

(b) By also allowing ε to tend to zero, show that $\frac{1}{2\pi i}\int_\gamma f(z)(\log z - \pi i)dz = \int_0^1 f(x)dx$, where $\gamma(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$.

(c) Use that observation to prove the bound $\left|\int_0^1 f(x)dx\right| \le \frac{1}{2}\int_0^{2\pi} \left|f(e^{i\theta})d\theta\right|$ (you can prove a strong inequality as well).

Solitions:

(a) Note that the integrands do in fact have the specified limits, and that they are uniformly continuous in the compact sets $[\varepsilon,1]\times[0,1]\cap\overline{D_1(0)}$, $[\varepsilon,1]\times[-1,0]\cap\overline{D_1(0)}$, respectively. Therefore the integrals converge accordingly.

(b) Consider the path along the inner part-circle α_δ, and along the full circle $\alpha(\theta) = \varepsilon e^{i\theta}$, $\theta\in[0,2\pi]$. For any $\varepsilon>0$, it's clear that $\lim_{\delta\to 0}\int_{\alpha_\delta} f(z)(\log(z)-\pi i)dz = \int_\alpha f(z)(\log(z)-\pi i)dz$. Moreover,

$$\begin{aligned}\left|\int_\alpha f(z)(\log(z)-\pi i)dz\right| &= \left|\int_\alpha f(z)\log(z)dz\right| = \\ &= \left|\int_\alpha f(z)(\ln\varepsilon + i\theta)dz\right| = \left|\ln\varepsilon\int_\alpha f(z)dz + i\int_\alpha f(z)\theta dz\right| \le \\ &\le 2\pi\varepsilon \sup_{|z|=\varepsilon}|f(z)|\ln\varepsilon + (2\pi)^2\varepsilon\sup_{|z|=\varepsilon}|f(z)| \underset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

Therefore we know that

$$\begin{aligned}\int_\gamma f(z)(\log z - \pi i)dz &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\left(\sum \mathrm{Res}(f,z) - \int_{AB+CD+\alpha_\delta} f(z)(\log z - \pi i)dz\right) = \\ &= \lim_{\varepsilon\to 0}\left(0 + 2\pi i\int_\varepsilon^1 f(x)dx - \int_\alpha f(z)(\log(z)-\pi i)dz\right) = \\ &= 2\pi i\int_0^1 f(x)dx\end{aligned}$$

(c) We simply note:

$$\left|\int_0^1 f(x)dx\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|f(e^{i\theta})\right|\left|e^{i\theta}(\pi - theta)\right|d\theta \le \frac{1}{2}\int_0^{2\pi}\left|f(e^{i\theta})\right|d\theta$$

9. By integrating along a keyhole curve, compute the following integrals:

   (a) $\int_0^\infty \frac{\ln(x)^2}{1+x+x^2}dx$
   Solitions:
   Here we also let the outer radius R of the keyhole curve to tend to infinity. By using the same method we did in the last exercise (first letting δ to tend to zero, then dealing with ε, R), we need only note that

$$\lim_{\delta\to 0}\int_{AB}\frac{\log^2 z}{z^2+z+1}dz = \int_\varepsilon^R \frac{(\ln x)^2}{x^2+x+1}dx$$
$$\lim_{\delta\to 0}\int_{CD}\frac{\log^2 z}{z^2+z+1}dz = -\int_\varepsilon^R \frac{(\ln x+2\pi i)^2}{x^2+x+1}dx$$

   Showing that the integrals along both circles vanish as $\varepsilon \to 0, R \to \infty$ is simple (bound the integrals as we did in the last part). We therefore know that

$$\frac{8\pi^3}{3\sqrt{3}} = 2\pi i\frac{(2\pi i/3)^2-(4\pi i/3)^2}{e^{2\pi i/3}-e^{-2\pi i/3}} = \int_0^\infty \frac{(\ln x)^2-(\ln x+2\pi i)^2}{x^2+x+1}dx =$$
$$= \int_0^\infty \frac{4\pi^2-4\pi i\ln x}{x^2+x+1}dx = 4\pi^2\int_0^\infty \frac{dx}{x^2+x+1} - 4\pi i\int_0^\infty \frac{\ln x}{x^2+x+1}dx$$

   This gives us the integrals $\int_0^\infty \frac{\ln x}{x^2+x+1}dx = 0$, $\int_0^\infty \frac{dx}{x^2+x+1} = \frac{2\pi}{3\sqrt{3}}$. It therefore stands to reason that repeating the process with $g(z) = \frac{\log^3 z}{z^2+z+1}$ will conclude the exercise, and indeed (again, showing that the relevant line integrals vanish is left as a simple exercise)

$$i\frac{7\cdot 16\pi^4}{27\sqrt{3}} = 2\pi i\frac{(2\pi i/3)^3-(4\pi i/3)^3}{e^{2\pi i/3}-e^{-2\pi i/3}} = \int_0^\infty \frac{(\ln x)^3-(\ln x+2\pi i)^3}{x^2+x+1}dx =$$
$$= \int_0^\infty \frac{-2\pi i(3\ln^2 x+6\pi i\ln x-4\pi^2)}{x^2+x+1}dx,$$

   which yields

$$6\pi i\int_0^\infty \frac{\ln(x)^2}{1+x+x^2}dx = 8\pi^3 i\int_0^\infty \frac{dx}{1+x+x^2} - i\frac{7\cdot 16\pi^4}{27\sqrt{3}} = \frac{32\pi^4 i}{27\sqrt{3}}$$

   (b) $\int_0^\infty \frac{\ln(1+x^2)}{x^{1+\alpha}}dx$, where $0<\alpha<2$ (hint: you might convert this integral into a more comfortable form)
   Solitions:
   We begin by integrating by parts: $\int_0^\infty \frac{\ln(1+x^2)}{x^{1+\alpha}}dx = 0 + 2\alpha^{-1}\int_0^\infty \frac{x^{1-\alpha}}{1+x^2}$. Hence, we might as well calculate $\int_0^\infty \frac{x^\beta}{x^2+1}dx$ for some $-1<\beta<1$. This conforms perfectly with the terms of the next part, so we'll use it for our solution.

   (c) $\int_0^\infty x^\alpha \frac{p(x)}{q(x)}dx$, where $-1<\alpha<1$, p, q are polynomials with $\deg q \geq 2+\deg p$, and q has no real zeroes. Here, the answer should naturally be given in terms of p, q.

Solitions:
Note that $\int_0^\infty x^\alpha \frac{p(x)}{q(x)}dx = \int_0^\infty e^{\alpha \ln x}\frac{p(x)}{q(x)}dx$. We can therefore utilize a keyhole curve to calculate the integral. First, note that the poles of $f(z) = e^{\alpha \ln z}\frac{p(z)}{q(z)}$, defined on $\mathbb{C}\setminus[0,\infty)$ (with the appropriate branch of logarithm), are the roots of q. We assumed that none of those are real, so they all are contained inside a keyhole curve, for sufficiently small ε and large R. The differentiation of z^α was handled in a previous exercise, and so the computations of residues is a simple matter, given p, q. In the special case that q only has simple roots, they are precisely $z_k^\alpha \frac{p(z_k)}{q'(z_k)}$. In any event, we note that

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{AB} f(z)dz = \int_\varepsilon^R f(x)dx\\
&\lim_{\delta\to 0}\int_{CD} f(z)dz = -e^{2\pi i\alpha}\int_\varepsilon^R f(x)dx\\
&\left|\int_{\gamma_R} f(z)dz\right| \le 2\pi R^{1+\alpha}\sup_{0\le\theta\le 2\pi}\frac{\left|p(Re^{i\theta})\right|}{\left|q(Re^{i\theta})\right|} \le K\frac{R^{1+\alpha+\deg p}}{R^{\deg q}} \xrightarrow[R\to\infty]{} 0\\
&\left|\int_{\gamma_\varepsilon} f(z)dz\right| \le 2\pi\varepsilon^{1+\alpha}\sup_{0\le\theta\le 2\pi}\left|\frac{p(\varepsilon e^{i\theta})}{q(\varepsilon e^{i\theta})}\right| \xrightarrow[\varepsilon\to 0]{} 0
\end{aligned}$$

It follows that $\int_0^\infty f(x)dx = (1-e^{2\pi i\alpha})^{-1}\sum_{q(z)=0}\text{Res}(f,z)$.

10. (a) How many roots of the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ lie inside the unit disc?
Solitions:
We let $f(z) = 6z^3$ and $g(z) = z^7 - 2z^5 - z + 1$. So $\min_{|z|=1}|f(z)| = 6 > 5 \ge \max_{|z|=1}|g(z)|$, hence $f, f+g$ have the same number of roots inside the unit disc (i.e., 3 roots).

(b) How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane?
Solitions:
Begin by denoting $f(z) = z^4 + 8z^3 + 3z^2 + 8z + 3$, and note that $\text{Re}\,(f(it)) = t^4 + 3(1-t^2) > 0$ for all $t \in \mathbb{R}$, and is even in t. In comparison, $\text{Im}\,(f(it)) = 8t(1-t^2)$ is odd in t.
We further note that for all $\varepsilon > 0$ there exists $R > 0$ such that for all $r \ge R$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ one has

$$\text{sgn}(\text{Re}\,(f(re^{i\theta}))) = \begin{cases} 1 & -\frac{\pi}{2} \le \theta \le -\frac{3\pi}{8} - \varepsilon \\ -1 & -\frac{3\pi}{8} + \varepsilon \le \theta \le -\frac{\pi}{8} - \varepsilon \\ 1 & -\frac{\pi}{8} + \varepsilon \le \theta \le \frac{\pi}{8} - \varepsilon \\ -1 & \frac{\pi}{8} + \varepsilon \le \theta \le \frac{3\pi}{8} - \varepsilon \\ 1 & \frac{3\pi}{8} + \varepsilon \le \theta \le \frac{\pi}{2} \end{cases}$$

and

$$\text{sgn}(\text{Im}\,(f(re^{i\theta}))) = \begin{cases} 1 & -\frac{\pi}{2} + \varepsilon \le \theta \le -\frac{\pi}{4} - \varepsilon \\ -1 & -\frac{\pi}{4} + \varepsilon \le \theta \le -\varepsilon \\ 1 & \varepsilon \le \theta \le \frac{\pi}{4} - \varepsilon \\ -1 & \frac{\pi}{4} + \varepsilon \le \theta \le \frac{\pi}{2} - \varepsilon \end{cases}$$

We therefore note that for γ_R a half circle in the right half plane, centered at the origin and with sufficiently large radius R, the image $f \circ \gamma$ is a contour which circles the origin twice, implying $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = 2$, which is also the number of zeroes f has in the right half plane.

(c) For how many z with $\mathrm{Im}\,(z) \geq 0$ is it true that $e^{iz} = z^2 + 2$?
Solitions:
We denote $f(z) = e^{iz}$,$g(z) = z^2 + 2$, and note that along the real axis $|f(t)| = 1 < 2 \leq 2 + t^2 = |g(t)|$. Moreover, for $r > \sqrt{3}$ and $0 \leq \theta \leq \pi$ it holds that $\left|f(re^{i\theta})\right| = e^{-R\sin\theta} \leq 1 < (r^2 - 2) \leq \left|g(re^{i\theta})\right|$. It follows that $f - g$ has the same number of zeroes in the upper half plane as g, namely one.

(d) For any $x > 1$, show that there exists a unique solution to the equation $e^{-z} = x - z$ in the right half plain, and that this solution is real.
Solitions:
We examine $f(z) = e^{-z}$,$g(z) = z - x$. On the imaginary axis, $|f(it)| = 1 < x \leq \sqrt{x^2 + t^2} = |g(z)|$. Moreover, for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $R > x + 1$, we have

$$\left|f(Re^{i\theta})\right| = e^{-R\cos\theta} \leq e^{-R} \leq R - x \leq \left|g(Re^{i\theta})\right|.$$

It follows that in any sufficiently large half circle, which rests symmetrically along the imaginary axis in the right half plane, $f(z)+g(z)$ has exactly one zero, because $g(z)$ has a unique zero at $z = x$.
Clearly, there exists a real solution, because $e^{-0} = 1$ and $e^{-t} \searrow 0$, whereas $x - 0 = x > 1$ and $x - t \searrow -\infty$, and both are continuous. Therefore, it's the sole solution.

11. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, and $f(a) = b$ for some $a \in \Omega$. Suppose $g(z) = f(z) - b$ has a zero of order 2 at a. Show that there exists a disk $D_r(b)$ and an open set $a \in U$ such that for any c in the punctured disk $D'_r(b)$, the equation $f(z) = c$ has two *distinct* roots in U.
Solitions:
Firstly, note that $f'(a) = 0$ but the zero is of finite order, so by the uniqueness theorem there exists $\varepsilon > 0$ such that $f'(z) \neq 0$ for all $z \in D'_\varepsilon(a) \subseteq \Omega$. We denote $f(z) = b + (z - a)^2 g(z)$ where g is holomorphic in Ω and $g(a) \neq 0$. It follows that there exists $0 < R < \varepsilon$ such that $|g(z)| \geq \frac{|g(a)|}{2}$ for all $z \in \overline{D_R(a)}$. We choose $r = R^2\frac{|g(a)|}{2}$, so that for all $c \in D_r(b)$, and $|z - a| = R$:

$$\left|(z - a)^2 g(z)\right| \geq R^2\frac{|g(a)|}{2}\,|b - c|\,,$$

Implying that $f(z) - c$ has the same number of zeroes in $D_R(a)$ as $(z - a)^2 g(z)$, namely two. However, f has no zeroes of order 2 in $D'_\varepsilon(a) \supseteq D'_R(a)$, implying that these two zeroes are distinct.

12. Let $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be defined as the angle between the segments $[0, x + iy]$ and $[x + iy, 1]$. Show that $u(x, y)$ is harmonic.
Solitions:
Then $u(x, y) = \arg\frac{(x-1)+iy}{x+iy}$. Note that $\frac{(x-1)+iy}{x+iy} = t \in \mathbb{R}$ if and only if $(1 - t)x - 1 = i(t - 1)y$, but since $y > 0$ it follows that $t = 1$, and $1 = 0$, in contradiction. Therefore we can use the principal branch of logarithm to denote $u(x, y) = \mathrm{Im}\left(\log\frac{(x+iy)-1}{x+iy}\right)$, and it follows that u is harmonic.